Exercise 44

Show by implicit differentiation that the tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Solution

Start by differentiating both sides of the given equation with respect to x.

$$\frac{d}{dx}\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = \frac{d}{dx}(1)$$

Use the chain rule to differentiate $y = y(x)$.

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$$

Solve for dy/dx.

$$\frac{2y}{b^2}\frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y}$$

The slope of the tangent line at the point (x_0, y_0) is then

$$m = -\frac{b^2}{a^2}\frac{x_0}{y_0}.$$

Use the point-slope formula to obtain the equation of the tangent line.

$$y - y_0 = m(x - x_0)$$

$$y - y_0 = -\frac{b^2}{a^2}\frac{x_0}{y_0}(x - x_0)$$

$$y_0 y - y_0^2 = -\frac{b^2}{a^2}x_0(x - x_0)$$

$$\frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} = -\frac{1}{a^2}x_0(x - x_0)$$

$$\frac{y_0 y}{b^2} - \frac{y_0^2}{b^2} = -\frac{x_0 x}{a^2} + \frac{x_0^2}{a^2}$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$$

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

The right side is 1 because the point (x_0, y_0) lies on the ellipse.